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Matrix Mean Series in Terms of Boundary Orthogonal Systems and Functions in the Classes H^∞ and E^p

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We give conditions for orthonormal systems on the boundary of a plane Jordan domain which are necessary and sufficient for an arbitrary series in terms of this orthonormal system to be the Fourier series of some function in $H^\infty(G)$ resp. $E^P(G)(1<p<\infty)$. Our results contain a classical criterion of Fejér for the boundedness of a holomorphic function in the unit disk.

1. INTRODUCTION

Let $H(\mathbb{D})$ denote the class of holomorphic functions in the open unit disk $\mathbb{D}$. For each $f \in H(\mathbb{D})$ we have a Taylor expansion $f(z) = \sum_{k=0}^{\infty} c_k(f) z^k$ with

$$c_k(f) = \frac{1}{2\pi i} \int_{|\zeta|=r<1} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta \qquad (k \in \mathbb{N}_0).$$

The following well-known criterion for the boundedness of a function in $H(\mathbb{D})$ goes back to Fejér [5, p. 22; 6, Chaps. III, IV]:

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THEOREM 1. *Let* $f \in H(\mathbb{D})$. *Then* $\sup_{z\in\mathbb{D}} |f(z)| < \infty$ *if and only if*

$$\sup_{n\in\mathbb{N}_0} \max_{|z|=1} \left| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) c_k(f) z^k \right| < \infty.$$

The aim of this paper is to generalize this results to series in terms of an orthonormal system on the boundary of a Jordan domain (cf. [10, p. 111; 1, p. 108; 6, Chap. IV, Sects. 1,2]).

2. THE MAIN RESULT

Let G be a Jordan domain with rectifiable boundary ∂G. We denote the length of ∂G by $|\partial G|$. A sequence of functions $\varphi_k : \partial G \to \mathbb{C}$ $(k \in \mathbb{N}_0)$ is said to be an orthonormal system on ∂G if

$$\frac{1}{|\partial G|} \int_{\partial G} \varphi_k(\zeta)\overline{\varphi_m(\zeta)}|d\zeta| = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases} \qquad (k, m \in \mathbb{N}_0). \tag{1}$$

For such an orthogonal system it follows that:

(a) $\varphi_k,\ \overline{\varphi_k} \in L^2(\partial G)$,

(b) $\varphi_k(\zeta) \neq 0$ almost everywhere on ∂G $(k \in \mathbb{N}_0)$.

The numbers

$$a_k := \frac{1}{|\partial G|} \int_{\partial G} f(\zeta)\overline{\varphi_k(\zeta)}|d\zeta| \qquad (k \in \mathbb{N}_0) \tag{2}$$

are called the *Fourier coefficients* of f, provided that the integrals exist. If all Fourier coefficients are defined, we call the formal series

$$f(z) \rightsquigarrow \sum_{k=0}^{\infty} a_k \varphi_k(z) \tag{3}$$

the *Fourier series* of f with respect to the orthonormal system (φ_k).

Here, it is not necessarily assumed that f itself is integrable, this is only required for $f(\zeta)\varphi_k(\zeta)$ on ∂G (cf. [11, pp. 48 and 185]). If for all $f \in L^p(\partial G)$ the Fourier series exists then $\varphi_k \in L^{\max\{2,p'\}}(\partial G)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

EXAMPLE. Let $G = \mathbb{D}$ and $\varphi_k(z) = z^k$, $a_k = c_k(f)$, $(k \in \mathbb{N}_0)$ (cf. [6, IV, Sects. 1,4]).

In the following, we consider an (infinite) matrix

$$M := (\mu_k^{(n)}) = \begin{pmatrix} \mu_0^{(0)} & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \mu_0^{(1)} & \mu_1^{(1)} & 0 & 0 & \cdots & \cdots & \cdots \\ \mu_0^{(2)} & \mu_1^{(2)} & \mu_2^{(2)} & 0 & \cdots & \cdots & \cdots \\ \mu_0^{(3)} & \mu_1^{(3)} & \mu_2^{(3)} & \mu_3^{(3)} & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \cdots \end{pmatrix}$$

of lower triangle shape.

THEOREM 2. *Let G be a simply connected Jordan domain with rectifiable boundary ∂G and M be a matrix as above with*

$$\lim_{n\to\infty} \mu_k^{(n)} = 1 \qquad (k \in \mathbb{N}_0). \tag{4}$$

Let (φ_k) be an orthonormal system with functions in $H^\infty(G)$ and assume that

$$k_1(M, \Phi) := \sup_{n\in\mathbb{N}_0} \sup_{z\in G} \frac{1}{|\partial G|} \int_{\partial G} \left| \sum_{k=0}^{n} \mu_k^{(n)} \varphi_k(z) \overline{\varphi_k(\zeta)} \right| |d\zeta| < \infty. \tag{5}$$

If $f \in H(G)$ then $\sup_{z\in G} |f(z)| < \infty$ if and only if

$$\sup_{n\in\mathbb{N}_0} \sup_{z\in G} \left| \sum_{k=0}^{n} \mu_k^{(n)} a_k \varphi_k(z) \right| < \infty. \tag{6}$$

Remark. In the case $G = \mathbb{D}$, we have (same notations as in the example above):

$$\sup_{|z|<1} \frac{1}{2\pi} \int_{|\zeta|=1} \left| \sum_{k=0}^{n} \mu_k^{(n)} \varphi_k(z) \overline{\varphi_k(\zeta)} \right| |d\zeta| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{n} \mu_k^{(n)} e^{ikt} \right| dt.$$

Let $\psi : \mathbb{D} \to G$ be a conformal map. By Γ_r we denote the image of $|w| = r$ under ψ. Then we have $|\Gamma_r| = r \int_{-\pi}^{\pi} |\psi'(re^{it})|\, dt$.

A function $f \in H(G)$ is said to be in the class $E^p(G)$ if

$$\sup_{r<1} \int_{\Gamma_r} |f(z)|^p |dz| < \infty. \tag{7}$$

Let G be a Jordan domain with smooth boundary and

(i) $s \to z(s)$ be the arc length parametrization of ∂G,

(ii) $s \to \theta(s)$ be the angle between the real axis and the tangent line to ∂G at the point $z(s)$,

(iii) $\delta \to \omega(\theta, \delta)$ be the continuity modulus of the function θ at the incline of the tangent line.

THEOREM 3. *Let G be a Jordan domain with smooth boundary satisfying*

$$\int_0^1 \frac{\omega(\theta,\delta)}{\delta} d\delta < \infty \tag{8}$$

and assume that $\varphi_k \in H^\infty(G)$. Moreover, let the matrix M and the system (φ_k) be as in Theorem 2, but with real coefficients of the matrix M. If $1 < p < \infty$ and $f \in H(G)$ then (7) holds if and only if

$$k_2^p := \sup_{n \in \mathbb{N}_0} \sup_{r<1} \frac{1}{|\Gamma_r|} \int_{\Gamma_r} \left| \sum_{k=0}^n \mu_k^{(n)} a_k \varphi_k(z) \right|^p |dz| < \infty. \tag{9}$$

3. PROOF OF THEOREM 2

We will need two lemmas. Because ∂G has a parametrization as a rectifiable curve each $f \in H^\infty(G)$ has a continuous extension on ∂G almost everywhere. If we define f to be 0 in the exceptional boundary points, we have $f \in L^\infty(\partial G)$.

LEMMA 1. *Let G be a Jordan domain with rectifiable boundary. Let M be a subdiagonal matrix and (φ_k) be an orthonormal system of functions in $H^\infty(G)$ and assume that (5) holds. Then for each $f \in H^\infty(G)$ the inequality*

$$\sup_{n \in \mathbb{N}_0} \sup_{z \in G} \left\| \sum_{k=0}^n \mu_k^{(n)} a_k \varphi_k(z) \right\| \leqslant k_1(M, \Phi) \sup_{z \in G} |f(z)|$$

is fulfilled.

Proof. A similar result for real orthogonal series has been proved in [4, Theorem [642] (necessity)]. By (2) we obtain on G

$$\sum_{k=0}^n \mu_k^{(n)} a_k \varphi_k(z) = \frac{1}{|\partial G|} \int_{\partial G} f(\zeta) \sum_{k=0}^n \mu_k^{(n)} \varphi_k(z) \overline{\varphi_k(\zeta)} |d\zeta| \qquad (n \in \mathbb{N}_0). \tag{10}$$

Obviously this implies

$$\sup_{z\in G}\left|\sum_{k=0}^{n}\mu_k^{(n)}a_k\varphi_k(z)\right|$$
$$\leqslant \sup_{\zeta\in G}|f(\zeta)|\sup_{z\in G}\frac{1}{|\partial G|}\int_{\partial G}\left|\sum_{k=0}^{n}\mu_k^{(n)}\varphi_k(z)\overline{\varphi_k(\zeta)}\right||d\zeta| \qquad (n\in\mathbb{N}_0).$$

Together with (5) this gives the desired result. ∎

Lemma 2. *Let G, ∂G, M and φ_k as in Theorem 2 and M be a subdiagonal matrix as indicated (but here (5) is not required) such that (4) is fulfilled.*

If the formal series $\sum_{k=0}^{\infty}a_k\varphi_k(z)$ with arbitrary complex coefficients a_k fulfills (6) then there exists some $f\in H^\infty(G)$ with (2) (i.e. the given series is the Fourier series of f).

Proof. The sequence $s_n(z) := \sum_{k=0}^{n}\mu_k^{(n)}a_k\varphi_k(z)\in H^\infty(G)$ is uniformly bounded in G by (6). Thus, a normal family argument gives some subsequence $(s_{n_j}(z))$ which converges locally uniformly in G to some function $f\in H^\infty(G)$.

Let Γ_r be defined as in Section 2. Then we have

$$\lim_{j\to\infty}\max_{z\in\Gamma_r}\left|f(z)-s_{n_j}(z)\right|=0$$

and therefore,

$$\lim_{j\to\infty}\max_{z\in\Gamma_r}\left|(f(z)-s_{n_j}(z))\overline{\varphi_k(z)}\right|=0 \qquad (k\in\mathbb{N}_0)$$

as well as

$$\lim_{j\to\infty}\int_{\Gamma_r}\left|(f(z)-s_{n_j}(z))\overline{\varphi_k(z)}\right||dz|=0 \qquad (r<1, k\in\mathbb{N}_0). \tag{11}$$

Because $f\in H^\infty(G)$ by construction we obtain that

$$(f(z)-s_n(z))\varphi_k(z)\in H^\infty(G)\subset E^1(G) \qquad (n,k\in\mathbb{N}_0).$$

A well-known result (cf. [6, Chap. IV; 3, Chap. X, Sect. 5(2)]) shows that

$$\lim_{r\to 1-0}\int_{\Gamma_r}|(f(z)-s_n(z))\varphi_k(z)||dz|$$
$$=\int_{\partial G}|(f(\zeta)-s_n(\zeta))\varphi_k(\zeta)||d\zeta| \qquad (n,k\in\mathbb{N}_0),$$

and thus

$$\lim_{r\to 1-0}\int_{\Gamma_r}\left|(f(z)-s_n(z))\overline{\varphi_k(z)}\right||dz|$$
$$=\int_{\partial G}\left|(f(\zeta)-s_n(\zeta))\overline{\varphi_k(\zeta)}\right||d\zeta| \qquad (n,k\in\mathbb{N}_0).$$

This means that for all $\varepsilon > 0$ there exists some $r_0 = r_0(k,\varepsilon) < 1$ such that for every $r \in [r_0, 1[$

$$\int_{\partial G}\left|(f(\zeta)-s_n(\zeta))\overline{\varphi_k(\zeta)}\right||d\zeta|$$
$$\leqslant \int_{\Gamma_r}\left|(f(z)-s_n(z))\overline{\varphi_k(z)}\right||dz|+\varepsilon \qquad (n,k\in\mathbb{N}_0).$$

This shows

$$\lim_{j\to\infty}\left|\int_{\partial G}(f(\zeta)-s_{n_j}(\zeta))\overline{\varphi_k(\zeta)}|d\zeta|\right| \leqslant \lim_{j\to\infty}\int_{\Gamma_r}\left|(f(z)-s_{n_j}(z))\overline{\varphi_k(z)}\right||dz|+\varepsilon$$
$$=\varepsilon \text{ by (11)} \qquad (k\in\mathbb{N}_0).$$

Since ε is an arbitrary positive number we obtain

$$\lim_{j\to\infty}\int_{\partial G}(f(\zeta)-s_{n_j}(\zeta))\overline{\varphi_k(\zeta)}|d\zeta| = 0 \qquad (k\in\mathbb{N}_0).$$

Now (1) shows that

$$\lim_{j\to\infty}\mu_k^{(n_j)}a_k = \frac{1}{|\partial G|}\int_{\partial G} f(\zeta)\overline{\varphi_k(\zeta)}|d\zeta| \qquad (k\in\mathbb{N}_0).$$

Assuming $j\to\infty$, we get the desired equation (2) from (4). ∎

4. PROOF OF THEOREM 3

We will need two lemmas. Because ∂G has a parametrization as a rectifiable curve each $f \in E^p(G)$ has continuous extension on ∂G almost everywhere. If we define f to be 0 in the exceptional boundary points, we have $f \in L^p(\partial G)$.

LEMMA 3. *Let G, ∂G, M, Φ as in Lemma 1, but with real coefficients of the matrix M. Then for each $f \in E^p(G)$ the inequality*

$$\sup_{n\in\mathbb{N}_0}\left\|\sum_{k=0}^{n} \mu_k^{(n)} a_k\varphi_k(z)\right\|_{L^p(\partial G)} \leqslant k_1(M,\Phi)||f||_{L^p(\partial G)}.$$

Proof. A similar result for real orthogonal series has been proved in [4, Theorem [641] (necessity)].

Equation (10) implies $(1=\frac{1}{p}+\frac{1}{p'})$ almost everywhere on ∂G

$$\left|\sum_{k=0}^{n} \mu_k^{(n)} a_k\varphi_k(z)\right| \leqslant \frac{1}{|\partial G|}\int_{\partial G} |f(\zeta)|^p \left|\sum_{k=0}^{n} \mu_k^{(n)}\varphi_k(z)\overline{\varphi_k(\zeta)}\right|^{1/p+1/p'} |d\zeta| \qquad (n\in\mathbb{N}_0).$$

Applying Hölder's inequality and by (5) we see

$$\left|\sum_{k=0}^{n} \mu_k^{(n)} a_k\varphi_k(z)\right| \leqslant \left(\frac{1}{|\partial G|}\int_{\partial G} |f(\zeta)|^p \left|\sum_{k=0}^{n} \mu_k^{(n)}\varphi_k(z)\overline{\varphi_k(\zeta)}\right||d\zeta|\right)^{1/p} k_1^{1/p'}(M,\Phi).$$

For $1<p<\infty$ we may write $(n\in\mathbb{N}_0)$

$$\left|\sum_{k=0}^{n} \mu_k^{(n)} a_k\varphi_k(z)\right|^p \leqslant k_1^{p-1}(M,\Phi)\,\frac{1}{|\partial G|}\int_{\partial G} |f(\zeta)|^p \left|\sum_{k=0}^{n} \mu_k^{(n)}\varphi_k(z)\overline{\varphi_k(\zeta)}\right||d\zeta|.$$

Because of $|w|=|\bar{w}|$ Fubini's Theorem leads to

$$\int_{\partial G}\left|\sum_{k=0}^{n} \mu_k^{(n)} a_k\varphi_k(z)\right|^p |dz| \leqslant k_1^{p-1}(M,\Phi)\frac{1}{|\partial G|}\int_{\partial G}|f(\zeta)|^p\left(\int_{\partial G}\left|\sum_{k=0}^{n} \overline{\mu_k^{(n)}}\varphi_k(z)\overline{\varphi_k(\zeta)}\right||dz|\right)|d\zeta| \quad (n\in\mathbb{N}_0).$$

From (5) we obtain

$$\int_{\partial G}\left|\sum_{k=0}^{n} \mu_k^{(n)} a_k\varphi_k(z)\right|^p |dz| \leqslant k_1^p(M,\Phi)\int_{\partial G}|f(\zeta)|^p|d\zeta| \qquad (n\in\mathbb{N}_0). \quad \blacksquare$$

LEMMA 4. *Let G, ∂G, M and φ_k as in Theorem 3 and M be a subdiagonal matrix as indicated (but here (5) is not required) such that (4) is fulfilled.*

If the formal series $\sum_{k=0}^{\infty} a_k \varphi_k(z)$ *with arbitrary complex coefficients* a_k *fulfills* (9) *then there exists some* $f \in E^p(G)$ *for some* $1<p<\infty$ *with* (2) (*i.e. the given series is the Fourier Series of* f).

Proof.
Step 1. Equation (9) implies

$$\sup_{n\in\mathbb{N}_0} \frac{1}{|\partial G|}\int_{\partial G} |s_n(\zeta)|^p |d\zeta| < \infty. \tag{12}$$

From (8) it follows that there exist constants k_3, k_4 such that [7, p. 48]

$$0 < k_3 \leqslant |\psi'(w)| \leqslant k_4 < \infty \qquad (|w| = 1). \tag{13}$$

By (12), (13) and $\sup_{n\in\mathbb{N}_0} \int_{-\pi}^{\pi} |s_n(\psi(e^{it}))|^p \, dt < \infty$ we obtain (see [2, 12.3.10(2)]) the existence of some $g \in L^p(\mathbb{T})$ and some subsequence (s_{n_j}) of (s_n) such that for all $u \in L^{p'}(\mathbb{T})$

$$\lim_{j\to\infty} \int_{-\pi}^{\pi} u(t) s_{n_j}(\psi(e^{it}))\, dt = \int_{-\pi}^{\pi} u(t) g(t)\, dt. \tag{14}$$

Step 2. From (14) and the Banach–Saks–Theorem (cf. [8, p. 90]) we can find a subsequence $(s_{n_{j_\nu}})$ of (s_{n_j}) such that

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} \left| g(t) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(\psi(e^{it})) \right|^p dt = 0.$$

By (13) we get

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} \left| g(t)\psi'(e^{it})e^{it} i - \frac{\psi'(e^{it})e^{it} i}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(\psi(e^{it})) \right|^p dt = 0. \tag{15}$$

Equation (9) implies that $s_n \in E^p(G)$ for all $n \in \mathbb{N}_0$ and by Cauchy's Integral Formula we see

$$\frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(z) = \frac{1}{2\pi(m+1)} \int_{-\pi}^{\pi} \frac{\psi'(e^{it})e^{it}}{\psi(e^{it}) - z} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(\psi(e^{it}))\, dt \qquad (z \in G).$$

The integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\psi'(e^{it})e^{it}}{\psi(e^{it}) - z} g(t)\, dt$$

defines a holomorphic function $f: G \to \mathbb{C}$ as well as a holomorphic function $h: \bar{\mathbb{C}} \backslash \bar{G} \to \mathbb{C}$ with $h(\infty) = 0$.

Now (15) implies (cf. [10, p. 107f])

$$\lim_{m\to\infty} \max_{z\in\Gamma_r} \left| f(z) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(z) \right| = 0 \qquad (r<1). \tag{16}$$

Minkowski's inequality gives by (9)

$$\left(\frac{1}{|\Gamma_r|} \int_{\Gamma_r} \left| \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(z) \right|^p |dz| \right)^{1/p} \leqslant k_2 \qquad (r<1, m \in \mathbb{N}_0). \tag{17}$$

Using (16) and (17) we obtain

$$\sup_{r<1} \left(\frac{1}{|\Gamma_r|} \int_{\Gamma_r} |f(z)|^p |dz| \right)^{1/p} \leqslant k_2$$

and this means that $f \in E^p(G)$.

Step 3. This step is analogously to the proof of Lemma 2.

Equation (9) gives $\varphi_k \in E^p(G) \subset E^1(G)$ for all $k \in \mathbb{N}_0$. Taking in account (16) this shows

$$\lim_{m\to\infty} \int_{\Gamma_r} \left| \left(f(z) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(z) \right) \overline{\varphi_k(z)} \right| |dz| = 0 \qquad (r<1, k \in \mathbb{N}_0). \tag{18}$$

At the end of the previous step, we obtained $f \in E^p(G) \subset E^1(G)$. Thus by (17) we see that

$$\left(f(z) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(z) \right) \varphi_k(z) \in E^p(G) \subset E^1(G) \qquad (m, k \in \mathbb{N}_0).$$

A well-known result [6, Chap. IV; 3, Chap. X, Sect. 5(2)] gives

$$\lim_{r\to 1-0} \int_{\Gamma_r} \left| \left(f(z) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(z) \right) \varphi_k(z) \right| |dz|$$
$$= \int_{\partial G} \left| \left(f(\zeta) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(\zeta) \right) \varphi_k(\zeta) \right| |d\zeta| \qquad (m, k \in \mathbb{N}_0).$$

Obviously this equation remains true if we replace φ_k by $\overline{\varphi_k}$. This means that for all $\varepsilon > 0$ there exists some $r_0 = r_0(k, \varepsilon) < 1$ such that for every

$r \in [r_0, 1[$

$$\int_{\partial G} \left| \left(f(\zeta) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(\zeta) \right) \overline{\varphi_k(\zeta)} \right| |d\zeta|$$
$$\leqslant \int_{\Gamma_r} \left| \left(f(z) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(z) \right) \overline{\varphi_k(z)} \right| |dz| + \varepsilon \qquad (m, k \in \mathbb{N}_0).$$

This shows

$$\lim_{m\to\infty} \left| \int_{\partial G} \left(f(\zeta) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(\zeta) \right) \overline{\varphi_k(\zeta)} |d\zeta| \right|$$
$$\leqslant \lim_{m\to\infty} \left| \int_{\Gamma_r} \left(f(z) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(z) \right) \overline{\varphi_k(z)} \right| |dz| + \varepsilon \overset{(18)}{=} \varepsilon \qquad (k \in \mathbb{N}_0).$$

Because $\varepsilon > 0$ is chosen arbitrarily we obtain

$$\lim_{m\to\infty} \int_{\partial G} \left(f(\zeta) - \frac{1}{m+1} \sum_{\nu=0}^{m} s_{n_{j_\nu}}(\zeta) \right) \overline{\varphi_k(\zeta)} |d\zeta| = 0 \qquad (k \in \mathbb{N}_0).$$

Since (φ_k) is an orthonormal system, we see that

$$\frac{1}{|\partial G|} \int_{\partial G} f(\zeta) \overline{\varphi_k(\zeta)} |d\zeta| = a_k \lim_{m\to\infty} \frac{1}{m+1} \sum_{\nu=0}^{m} \mu_k^{(n_{j_\nu})} \qquad (k \in \mathbb{N}_0).$$

Now (4) shows the desired equation (2) and Lemma 4 is proved. ∎

5. PROBLEM

It would be interesting to replace (1) by

$$\frac{1}{|\partial G|} \int_{\partial G} n(\zeta) p_k(\zeta) \overline{p_m(\zeta)} |d\zeta| = \begin{cases} 0 & \text{if } k \neq m, \\ 1 & \text{if } k = m, \end{cases}$$

where $n(\zeta)$ is some weight function and the p_m are suitable polynomials as studies by Szegö (cf. [9, p. 364]).

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